$L_p \to L_q$ - ESTIMATES FOR SOME POTENTIAL-TYPE OPERATORS WITH OSCILLATING KERNELS

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Abstract

We obtain $L_p \to L_q$-estimates for the potential operator K_a^α in R^n with a radial kernel of the form $a(|t|)e^{i|t|}/|t|^{n-\alpha}$, $0 < \alpha < n$, where $a(\infty) \neq 0$ and the characteristic $a(r)$, is sufficiently smooth in some neighborhood of infinity and locally satisfies some general assumptions. In particular, $a(r)$ may have power singularities. We construct some convex sets on the $(1/p, 1/q)$-plane for which the operator K_a^α is bounded from L_p into L_q, as well as the domains where K_a^α is not bounded.

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1. Introduction

We give the $L_p \to L_q$-estimates for the potential–type operator

$$(K_a^\alpha \varphi)(x) = \int\limits_{\mathrm{R}^n} \frac{a(|t|)e^{i|t|}}{|t|^{n-\alpha}} \varphi(x-t)\, dt, \quad 0 < \alpha < n, \qquad a(\infty) \neq 0. \tag{1.1}$$

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The characteristic $a(r)$ is assumed to be sufficiently smooth in some neighborhood of infinity (for $r > A$) and locally satisfies some general assumptions. Namely,

$$\operatorname{mes}\left\{t: \ |k_a^{\alpha,0}(|t|)| > \lambda\right\} \le c\lambda^{-\frac{n}{n-\alpha}}, \quad \lambda > 0, \tag{1.2}$$

where $k_a^{\alpha,0}(|t|) = \chi_A(t)a(|t|)e^{i|t|}/|t|^{n-\alpha}$, $\chi_A(t)$ being the characteristic function of the ball $|t| \le A$. Condition (1.2) covers, in particular, characteristics $a(r)$ with power singularities at some points $r_1, \dots, r_m$ of orders $\beta_k - 1$, where $\beta_k \ge \alpha/n$, $1 \le k \le m$.

We construct some convex sets on the $(1/p, 1/q)$-plane for which K_a^α is bounded from L_p into L_q as well as the domains, where K_a^α is not bounded. In some cases we precisely construct the $\mathcal{L}$-characteristic $\mathcal{L}(K_a^\alpha)$ of the operator K_a^α.

In the case, when $0 < \beta_{k_0} < \alpha/n$ for some k_0, the condition (1.2) is violated. This fact essentially influences on the picture of boundedness of the operator K_a^α. To illustrate this effect, we consider a characteristic of the form

$$a(r) = b(r) + \chi(r)d(r)(1 - r + i0)^{\beta-1}, \quad 0 < \beta < 1, \tag{1.3}$$

where $b(r)$ satisfies (1.2), $d(r) \in C^m(1/2, 3/2)$, $m \ge \max\{3, [n/2] + 1\}$ and $d(1) \ne 0$, $\chi(r)$ is the characteristic function of the interval $(1/2, 3/2)$. We show, that the sets $\mathcal{L}(K_a^\alpha)$ and $\mathcal{L}(K_b^\alpha)$ may essentially differ from each other in the case $\beta < \alpha/n$. More exactly, the set $\mathcal{L}(K_b^\alpha)$ may be considerably wider (see Remark 2.3).

Let us justify our interest to $L_p \to L_q$-estimates for the operator (1.1). First, we consider the generalized Riesz potential with the kernel $a(|t|)/|t|^{n-\alpha}$, investigated in [14] (see also the books [11], Subsection 8.5 and [12, p. xxi]), where $a(r)$ is in the class $C^m(\dot{\mathrm{R}}_+^1)$, $m > [\alpha] + 1$. It is easy to show, that this potential is bounded from L_p into L_q if and only if the point $(1/p, 1/q)$ belongs to the interval $\alpha/n < 1/p < 1$, $1/q = 1/p - \alpha/n$. In contrast to this case, the operator (1.1) is bounded from L_p into L_q for the points $(1/p, 1/q)$, belonging to some convex sets of positive Lebesgue measure. Such an effect is stipulated by the oscillating exponent in the kernel of this operator.

On the other hand, this interest is caused by usage of $L_p \to L_q$-estimates in problems of inversion and characterization of potentials (1.1) with densities in L_p. Such an application will be given in another paper. Here we only refer to the surveying paper [8] (see also [6]) for the case of fractional Strichartz potentials and their modifications. (These potentials are close to the operators (1.1) in the sense that their kernels have singularities on a sphere.)

We note that the investigations on the $L_p \to L_q$-estimates for potentials with oscillating kernels are at the very beginning. Here we can mention only two cases of "specific" oscillation, generated by the Bessel function (the Bochner–Riesz operators, see [1]), the Hankel function (the acoustic potentials, see [7]), and the case when $a(r) \equiv 1$ in (1.1) (see [9], [8]). We also note that there are several papers on the L_p-estimates for the Bochner–Riesz operator, which gave rise to further investigations of potentials with oscillating kernels (we refer to the book [16], Ch. 9, § 6 for surveying the results of these papers).

The paper is organized as follows. In Section 2 we formulate our main results (Theorems 2.1 and 2.2) and give some comments to them. Section 3 contains the necessary preliminaries. In Section 4 we prove some auxiliary statement. Section 5 is devoted to the proof of Theorems 2.1 and 2.2.

2. The main results and some comments

We use the following notation to formulate our main results:

$a = 1 - \frac{\alpha}{n}, \quad b = \frac{2n+\alpha(n-1)}{n(n+1)}, \quad d = \frac{\alpha+1}{n+1},$

$e = 1 - \frac{(n-\alpha)(n-1)}{n(n+3)}, \quad k = \frac{n+3}{2(n+1)},$

$A' = (1-a, 0), \quad B' = (1-a, 1-b), \quad C' = (1-k, 1-k),$

$G' = (1-a, 1-e), \quad H' = (1-a, 1-a), \quad A = (1, a), \quad B = (b, a),$

$C = (k, k), \quad G = (e, a), \quad H = (a, a),$

$D = (d, 1-d), \quad O' = (0,0), \quad O = (1,1), \quad E = (1,0), \quad F = (1/2, 1/2);$

$(A, B, C, \dots, K)$ is an open polygon with the vertices at the points $A, B, C, \dots, K$;

$[A, B, C, \dots, K]$ is its closure.

We introduce the sets:

$$\mathcal{L}_1(\alpha,n)=\begin{cases}(A',G',D,G,A,E)\cup(A',E)\cup(A,E]\cup\{D\}, & \\ \qquad n\geq 3,\ \frac{n-1}{2}<\alpha<n\ , & \\ (A',G',G,A,E)\cup(A',E)\cup(A,E], & \\ \qquad n\geq 3,\ \frac{n(n-1)}{2(n+1)}\leq\alpha<\frac{n-1}{2}\ , & \\ (A',G',F,G,A,E)\cup(A',E)\cup(A,E], & \\ \qquad n\geq 3,\ \alpha=\frac{n-1}{2}\ , & \\ [A',C',C,A,E]\setminus(\{A'\}\cup\{A\}), & \\ \qquad n>3,\ \frac{n-1}{4}\leq\alpha<\frac{n(n-1)}{2(n+1)}\ , & \\ [A',H',H,A,E]\setminus([A',H']\cup[A,H]), & \\ n=2,\ 0<\alpha<1/2,\quad \text{or}\quad n=3,\ 0<\alpha<3/4\ , & \\ \qquad \text{or}\quad n>3,\ 0<\alpha<\frac{n-1}{4}\ , & \\ (A',B',B,A,E)\cup(A',E)\cup(A,E]\cup\{D\}, & \\ \qquad n=2,\ 1/2<\alpha<2\ , & \\ (A',B',B,A,E)\cup(A',E)\cup(A,E], & \\ \qquad n=2,\ \alpha=1/2\ ; & \end{cases}$$

$$\mathcal{L}_2(\alpha,n)=\begin{cases}\mathcal{L}_1(\alpha,n)\cup(F,G,G')\cup(G',G)\cup\{F\}, & \text{if}\quad \alpha\in\left[\frac{n(n-1)}{2(n+1)},\frac{n-1}{2}\right)\\ \mathcal{L}_1(\alpha,n), \quad \text{if}\quad \alpha\notin\left[\frac{n(n-1)}{2(n+1)},\frac{n-1}{2}\right) & \end{cases}$$

(see **Pictures 1 and 2**).

It is easily verified that

$$\mathcal{L}_2(\alpha,n)\subset\mathcal{L}_2(\alpha-1,n),\quad 1<\alpha<n. \tag{2.1}$$

By $\mathcal{L}(A)$ we denote the $\mathcal{L}$-characteristic of the operator A, that is, the set of all pairs $(1/p,1/q)$ for which A is bounded from L_p into L_q.

Let $a(r)$ be such that:

1) the function $a^*(r)=a(r^{-1})$, $r>0$, $a^*(0)=\lim\limits_{r\to 0}a(r^{-1})$ is continuously differentiable up to the order $[\alpha]+2$ on the interval $[0,A^{-1})$ for some $A>0$, and $a^*(0)=a(\infty)\neq 0$;

2) the condition (1.2) is fulfilled.

The case of such characteristics is covered by the following theorem.

THEOREM 2.1. *Let $0 < \alpha < n$ and $a(r)$ satisfy conditions 1) and 2).*
I. The imbedding

$$\mathcal{L}(K_a^\alpha) \supset (\mathcal{L}_2(\alpha, n) \setminus [A', A, E]) \cup (A', A) \tag{2.2}$$

is valid.
II. The set $\mathcal{L}(K_a^\alpha)$ does not contain the points lying:

1) *on the segment $[A, H]$ and above it;*
2) *on the segment $[A', H']$ and to the left of it;*
3) *above the straight line $B'B$, if $(n-1)/2 < \alpha < n$;*
4) *on the segment $[O', O]$, if $\alpha = (n-1)/2$;*
5) *below the straight line $A'A$ in the case, when $a(r)$ stabilizes at the origin as a Hölder function, that is,*

$$|a(r) - a(0)| \le c \cdot r^\lambda, \quad 0 < r < \eta < A, \tag{2.3}$$

for some $\lambda > 0$.

REMARK 2.1. We observe that

$$\mathcal{L}(K_a^\alpha) = (\mathcal{L}_2(\alpha, n) \setminus [A', A, E]) \cup (A', A), \quad \text{if} \quad n = 2 \quad \text{and} \quad 0 < \alpha \le 1/2,$$

or $n = 3$ and $0 < \alpha < 3/4$, or $n > 3$ and $0 < \alpha < (n-1)/4$.

We consider below a characteristic of the form (1.3), where $b(r)$ satisfies the assumptions of Theorem 2.1.

As is easily seen, the condition (1.2) holds for $a(r)$, if $\alpha/n \le \beta < 1$. Therefore, the statements of Theorem 2.1 are valid for the operator K_a^α. This condition is violated if $0 < \beta < \alpha/n$ (see Remark 5.1). In the last case the sets $\mathcal{L}(K_a^\alpha)$ and $\mathcal{L}(K_b^\alpha)$ may essentially differ from each other. This is seen from the next theorem, which provides $L_p \to L_q$-estimates for the operator K_a^α, where $a(r)$ has the form (1.3) (see also **Pictures 1-4**). We denote

$$T' = (\beta, 0), \quad T = (1, 1-\beta), \quad P' = \left(\frac{n + 2\beta - 1}{n+1}, 0\right),$$
$$P = \left(1, \frac{2(1+\beta)}{n+1}\right), \quad M = \left(\frac{n+\beta}{n+1}, \frac{1-\beta}{n+1}\right),$$

$$\Omega(\alpha,\beta,n) = \Big(\Big(\mathcal{L}_2(\alpha,n) \cap \Big([O', O, T, M, T'] \setminus (\{T'\} \cup \{T\})\Big)\Big) \setminus \setminus [A', A, E]\Big) \cup (A', A).$$

THEOREM 2.2. *Let $0 < \alpha < n$, $0 < \beta < \alpha/n$.*

I. The imbedding

$$\mathcal{L}(K_a^\alpha) \supset \Omega(\alpha,\beta,n)$$

is valid.

II. The set K_a^α does not contain the points lying:

1) on the segment $[A, H]$ and above it;

2) on the segment $[A', H']$ and to the left of it;

3) above the straight line $B'B$, if $(n-1)/2 < \alpha < n$, $\beta \geq \alpha + 1 - n$ (for $\alpha \leq n-1$ this inequality is valid for every β, $0 < \beta < \alpha/n$);

4) below the straight lines MT and MT', if $0 < \alpha < n$, $\beta > \alpha + 1 - n$;

5) below the straight line $A'A$, when $b(r)$ satisfies (2.3), if $0 < \alpha < n$, $\beta \geq \alpha + 1 - n$;

6) outside the interval (B', B), if $n - 1 < \alpha < n$, $\beta = \alpha + 1 - n$;

7) on the straight line $P'P$ and above it and also on the straight lines $B'B$, $A'A$ and between them, if $n - 1 < \alpha < n$, $\beta < \alpha + 1 - n$;

8) on the segment $[O', O]$, if $\alpha = (n-1)/2$.

REMARK 2.2. We observe that $\mathcal{L}(K_a^\alpha) = \Omega(\alpha,\beta,n)$, if $n = 2$ and $0 < \alpha \leq 1/2$ or $n = 3$ and $0 < \alpha < 3/4$, or $n > 3$ and $0 < \alpha < (n-1)/4$.

REMARK 2.3. Here we give two examples of essential differences between $\mathcal{L}(K_a^\alpha)$ and $\mathcal{L}(K_b^\alpha)$.

1. Let $n - 1 < \alpha < n$, $\beta = \alpha + 1 - n$. As is seen from the statements I and II (item 6)), the set $\mathcal{L}(K_a^\alpha)$ contains the point D and does not contain

the points of the set $V = (D, G', A', A, G) \cup (A', A)$, while $V \cup \{D\} \subset \mathcal{L}(K_b^\alpha)$ (see **Picture 3**).

2. Let $n-1 < \alpha < n$, $0 < \beta < \alpha + 1 - n$. In accordance with the statement II (item 7)), $\mathcal{L}(K_a^\alpha)$ does not contain the set $V \cup \{D\}$, while this set is imbedded into $\mathcal{L}(K_b^\alpha)$ (see **Picture 4**). Moreover,

$$\mathcal{L}(K_a^\alpha) \cap \mathcal{L}(K_b^\alpha) = \emptyset,$$

if $b(r)$ satisfies (2.3).

3. Preliminaries

3.1. Notation: S^{n-1} is the unit sphere in R^n; $L_p = L_p(\mathrm{R}^n)$; $\|f\|_p = \|f\|_{L_p}$; $(Ff)(\xi) = \displaystyle\int_{\mathrm{R}^n} f(x) e^{i\xi x}\, dx$ is the Fourier transform of a function f; S is the Schwartz class of rapidly decreasing smooth functions; $\mathcal{R}_0 = \{\varphi : \ \varphi = Ff, \ f \in L_1\}$ is the Wiener ring. Following [3], we denote by L_p^q the space of distributions $k \in S'$ such that $\|k * f\|_q \leq C\|f\|_p$, $f \in S$, the constant $C > 0$ not depending on f; the Fourier dual space $F(L_p^q)$ is denoted by M_p^q. The end of proof is denoted by $\square$.

3.2. $L_p \to L_q$**-estimates for the operator (1.1) in the model case** $a(r) \equiv 1$**.**

In the case $a(r) \equiv 1$ we write $K_a^\alpha \equiv K^\alpha$.

THEOREM 3.1. (see [9]; [8], Theorem 3.1) *Let* $0 < \alpha < n$.

I. The imbedding

$$\mathcal{L}(K^\alpha) \supset (\mathcal{L}_1(\alpha, n) \setminus [A', A, E]) \cup (A', A)$$

is valid.

II. The set $\mathcal{L}(K^\alpha)$ *does not contain the points lying:*

1) *on the segment* $[A, H]$ *and above it;*

2) *on the segment* $[A', H']$ *and to the left of it;*

3) *above the straight line* $B'B$ *in the case* $(n-1)/2 < \alpha < n$*;*

4) *below the straight line* $A'A$

(see **Pictures 1 and 2***).*

3.3. On some oscillating $p \to q$-multipliers.

Let $m_b^{\pm}(\xi) = u(|\xi|^2)|\xi|^{-b}e^{\pm i|\xi|}$, $b > 0$, where $u(r) \in C^{\infty}(\mathrm{R}^1_+)$, $0 \le u(r) \le 1$, $u(r) = 0$ for $r \le 1$ and $u(r) = 1$ for $r \ge 2$.

THEOREM 3.2. ([5]) *The following relations are valid:*

1) $m_b^{\pm} \in M_p^q$, $p \le q$ *if and only if* $1/p + 1/q \le 1$, $1/p - n/q \le b - (n-1)/2$ *or* $1/p + 1/q \ge 1$, $n/p - 1/q \le b + (n-1)/2$;

2) $m_b^{\pm} \in M_1^q$, $1 < q < \infty$ *if and only if* $b - (n+1)/2 > -1/q$;

3) $m_b^{\pm} \in M_p^{\infty}$, $1 < p < \infty$ *if and only if* $1/p < b - (n-1)/2$;

4) $m_b^{\pm} \in M_1^{\infty}$ *if and only if* $b > (n+1)/2$.

3.4. Uniform asymptotic expansion for the Bessel function $J_\nu(z)$.

Let $\Omega = \{z \in \mathrm{C}: \ |z| > \eta, \ |\arg z| < \theta\}$, where $\eta > 0$ and $\theta \in (0, \pi/2)$. Representing $J_\nu(z)$ as a linear combination of the Hankel functions $H^{(1)}_{\pm\nu}(z)$ and $H^{(2)}_{\pm\nu}(z)$ (we take $+\nu$ if $\nu > -1/2$ and $-\nu$ otherwise) and applying the results of [17, p. 220], we arrive at the equality

$$J_\nu(z) = \left(\frac{\pi z}{2}\right)^{-1/2}\left[e^{-iz}\left(\sum_{m=0}^{N} C^{(\nu)}_{m,-}z^{-m} + R^{(\nu)}_{N,-}(z)\right) + \right. \\ \left. + e^{iz}\left(\sum_{m=0}^{N} C^{(\nu)}_{m,+}z^{-m} + R^{(\nu)}_{N,+}(z)\right)\right], \tag{3.1}$$

where $C^{(\nu)}_{0,\pm} = \frac{1}{2}e^{\mp(i\pi/4)(2\nu+1)}$.

REMARK 3.1. The remainders $R^{(\nu)}_{N,\pm}(z)$ are analytic in Ω and $R^{(\nu)}_{N,\pm}(z) = O(|z|^{-N-1})$, as $|z| \to \infty$ and $(d/dz)^j R^{(\nu)}_{N,\pm}(z) = O(|z|^{-N-1-j})$, $|z| \to \infty$ in any closed sector $\Omega_0 \subset \Omega$ (see [10, p. 21].

3.5. Asymptotic expansion of certain integrals, containing an oscillating exponent.

Direct analysis of the proof of the Erdélyi lemma given in [2, p. 95] shows that the following lemma is valid.

LEMMA 3.1. *Let $\beta > 0$, $f(x) \in C^m([0,a])$, $m = 2N + 1 + [\beta]$ and $f^{(j)}(a) = 0$ $(j = 0, 1, \dots, m)$. Then*

$$\int\limits_0^a x^{\beta-1} f(x) e^{\pm i\lambda x}\, dx = \sum_{k=0}^{N-1} a^{\pm}_{k,\beta} \lambda^{-(k+\beta)} + W^{\pm,\beta}_N(\lambda), \tag{3.2}$$

where

$$a^{\pm}_{k,\beta} = \frac{f^{(k)}(0)}{k!} (\pm i)^{\beta+k} \Gamma(\beta + k) \quad \text{and}$$

$$|(W^{\pm,\beta}_N(\lambda))^{(j)}| \le \frac{C^{\pm,j}}{\lambda^{N+\beta+j}}, \quad j = 0, 1, \dots \quad \text{for any } \lambda > 1, \tag{3.3}$$

the constants $C^{\pm,j}$ not depending on λ.

COROLLARY 3.1. *Let the conditions of Lemma 3.1 be fulfilled for $N = 1$. Then $|\xi|^{\beta+1} W^{\pm,\beta}_1(|\xi|) \in M^p_p$, $1 < p < \infty$.*

The statement of Corollary 3.1 is derived from (3.3) by the direct application of the well-known Mikhlin theorem (see [15, p. 152]).

We also need the following lemma.

Let $\theta(s) \in C^\infty(\mathrm{R}^1_+)$ be such that $0 \le \theta(s) \le 1$, $\theta(s) = 0$, if $s \le 1$ and $\theta(s) = 1$, if $s \ge 2$.

LEMMA 3.2. ([4]) *Let $\lambda \in \mathrm{C}$. For $\varepsilon > 0$, set*

$$J_{\lambda,\varepsilon}(t) = (2\pi)^{-n} \int\limits_0^\infty \theta(s) s^\lambda e^{ist-\varepsilon s}\, ds.$$

Then $J_{\lambda,\varepsilon}(t)$ converges, as ε tends to zero, uniformly in $|t| \ge \delta$ for every $\delta > 0$, and the resulting function $J_\lambda(t) = \lim\limits_{\varepsilon\to 0} J_{\lambda,\varepsilon}(t)$ has the following estimates:

$$J_\lambda(t) = \begin{cases} A_\lambda (t + i0)^{-\lambda-1} + \tilde{J}_\lambda(t), & \lambda \ne -1, -2, \dots, \\ A'_\lambda t^{-\lambda-1} + A''_\lambda t^{-\lambda-1} \ln(t + i0) + \tilde{J}_\lambda(t), & \lambda = -1, -2, \dots, \end{cases}$$

where $\tilde{J}_\lambda(t)$ is smooth on R and A_λ, A'_λ and A''_λ are constants depending only on λ $\Big(A_\lambda = (2\pi)^{-n} e^{\frac{i\pi}{2}(1+\lambda)} \Gamma(1+\lambda)$, $A'_\lambda = (2\pi)^{-n} \dfrac{\psi(-\lambda) + i\pi/2}{(-\lambda - 1)!} e^{\frac{i\pi}{2}(-\lambda-1)}$,

$A''_\lambda = -(2\pi)^{-n} \dfrac{e^{\frac{i\pi}{2}(-\lambda-1)}}{(-\lambda - 1)!}$, $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}\Big)$ *and*

$$J_\lambda(t) = O(|t|^{-M}) \quad \text{as} \quad t \to \infty \quad \text{for every} \quad M > 0.$$

We note, that Lemma 3.2 provides the asymptotic behaviour of $J_\lambda(t)$, as $t \to 0$.

4. Some auxiliary statement

We need the following theorem (see also **Pictures 1 and 2**), providing $L_p \to L_q$-estimates for the operator

$$(S^\alpha \varphi)(x) = \int\limits_{|t|>A} \frac{e^{i|t|}}{|t|^{n-\alpha}} \varphi(x-t)\, dt. \tag{4.1}$$

THEOREM 4.1. *Let $0 < \alpha < n$.*

I. The imbedding

$$\mathcal{L}(S^\alpha) \supset \mathcal{L}_2(\alpha, n). \tag{4.2}$$

is valid.

II. The set $\mathcal{L}(S^\alpha)$ does not contain the points, lying:

1) *on the segment $[A, H]$ and above it;*

2) *on the segment $[A', H']$ and to the left of it;*

3) *above the straight line $B'B$ in the case $(n-1)/2 < \alpha < n$;*

4) *on the segment $[O', O]$, if $\alpha = (n-1)/2$.*

P r o o f. In the case $\alpha \notin \left[\frac{n(n-1)}{2(n+1)}, \frac{n-1}{2}\right)$, the imbedding (4.2) follows from Theorem 3.1 due to the following facts:

1) As is easily seen, the $\mathcal{L}$-characteristic of the operator

$$(H^\alpha \varphi)(x) = \int\limits_{|t|<A} \frac{e^{i|t|}}{|t|^{n-\alpha}} \varphi(x-t)\, dt,$$

has the form

$$\mathcal{L}(H^\alpha) = [O', A', A, O] \setminus (\{A'\} \cup \{A\}).$$

2) The kernel of S^α belongs to L_q for all $q > n/(n-\alpha)$.

Passing to the case $\alpha \in \left[\frac{n(n-1)}{2(n+1)}, \frac{n-1}{2}\right)$, we prove first that S^α is bounded in L_2 for $\alpha < (n-1)/2$. With the aid of Lemma 3.2 it is not difficult to obtain the following representation for the symbol $m_\alpha(|\xi|)$ of the operator S^α in some neighbourhood of the unit sphere:

$$m_\alpha(|\xi|) = C_\alpha(1-|\xi|+i0)^{\frac{n-1}{2}-\alpha} + U(|\xi|), \quad \text{if} \quad \alpha - \frac{n+1}{2} \neq -1, -2, \dots,$$

$$\begin{aligned} m_\alpha(|\xi|) \;&=\; C'_\alpha(1-|\xi|)^{\frac{n-1}{2}-\alpha} + C''_\alpha(1-|\xi|)^{\frac{n-1}{2}-\alpha}\ln(1-|\xi|+i0)+ \\ &\quad +V(|\xi|), \quad \text{if} \;\; \alpha - \frac{n+1}{2} = -1, -2, \dots, \end{aligned} \tag{4.3}$$

where $U(|\xi|)(V(|\xi|)) = o(|1-|\xi||^{\frac{n-1}{2}-\alpha})$, as $|\xi| \to 1$,

$$\begin{aligned} C_\alpha &= (2\pi)^{-n} e^{\frac{i\pi}{2}(\alpha - \frac{n-1}{2})}\Gamma\left(\alpha - \frac{n-1}{2}\right), \\ C'_\alpha &= (2\pi)^{-n}\frac{\psi\left(\frac{n+1}{2}-\alpha\right)+i\frac{\pi}{2}}{\left(\frac{n-1}{2}-\alpha\right)!} e^{\frac{i\pi}{2}\left(\frac{n-1}{2}-\alpha\right)}, \\ C''_\alpha &= -(2\pi)^{-n}\frac{e^{\frac{i\pi}{2}\left(\frac{n-1}{2}-\alpha\right)}}{\left(\frac{n-1}{2}-\alpha\right)!}. \end{aligned}$$

Moreover, $m_\alpha(|\xi|)$ is bounded outside of the mentioned neighbourhood. From here we derive that $m_\alpha \in M_2^2$, hence, S^α is bounded in L_2. Applying the convexity arguments, we arrive at (4.2) in the case $\alpha \in \left[\frac{n(n-1)}{2(n+1)}, \frac{n-1}{2}\right)$.

Let us prove the statement II. To prove 1) we consider the characteristic function $\tilde{\chi}(y)$ of the ball $|y| < 1/10$; $\tilde{\chi}(y) \in L_p$, $1 \le p \le \infty$. Evaluating the integral

$$(S^\alpha \tilde{\chi})(x) = \int\limits_{|t|<1/10} \frac{e^{i|x-t|}}{|x-t|^{n-\alpha}}\, dt,$$

it is easy to show that $|(S^\alpha\tilde{\chi})(x)| \ge \frac{C}{|x|^{n-\alpha}}$, as $|x| \to \infty$. Hence, $S^\alpha\tilde{\chi} \notin L_{\frac{n}{n-\alpha}}$. This means that the points of the segment $[A, H]$ do not belong to $\mathcal{L}(S^\alpha)$. By convexity arguments we obtain 1). Then 2) follows from 1) by duality.

The statement 3) follows from the correspondent statement of Theorem 3.1.

Since the operator $S^{\frac{n-1}{2}}$ is not bounded in L_2, because $m_{\frac{n-1}{2}} \notin M_2^2$ (as is seen from (4.3)), we obtain 4).

■

5. Proof of the main results

5.1. Proof of Theorem 2.1.

We represent the potential $K_a^\alpha \varphi$ as follows

$$(K_a^\alpha \varphi)(x) = \left(\int\limits_{|t|<A} + \int\limits_{|t|>A} \right) \frac{a(|t|)e^{i|t|}}{|t|^{n-\alpha}} \varphi(x-t)\, dt$$
$$\equiv (K_a^{\alpha,0}\varphi)(x) + (K_a^{\alpha,\infty}\varphi)(x). \tag{5.1}$$

The proof will be based on (5.1) and the following lemmas.

Lemma 5.1. *The imbedding*

$$\mathcal{L}(K_a^{\alpha,0}) \supset [O', O, A, A'] \setminus (\{A'\} \cup \{A\}) \tag{5.2}$$

is valid. Besides this, the set $\mathcal{L}(K_a^{\alpha,0})$ does not contain the points of the set $[A', A, E] \setminus (A', A)$, if $a(r)$ satisfies the condition (2.3).

P r o o f. It is evident that the kernel $k_a^{\alpha,0}(|t|)$ of the operator $K_a^{\alpha,0}$ belongs to L_1. Indeed,

$$\|k_a^{\alpha,0}\|_1 = \int\limits_0^\infty \lambda_{k_a^{\alpha,0}}(t)\, dt \le C + \int\limits_1^\infty \frac{dt}{t^{n/(n-\alpha)}} < \infty.$$

Therefore, $K_a^{\alpha,0}$ is bounded in L_p, $1 \le p \le \infty$.

On the other hand, the statement of Sobolev theorem is valid for $K_a^{\alpha,0}$. Namely, this operator is bounded from L_p into L_q, $1 < p < n/\alpha$, $q = np/(n-\alpha p)$ and weakly bounded from L_1 into $L_{\frac{n}{n-\alpha}}$ (see [15], the remark on the page 142). Hence (5.2) follows.

To prove the second statement of lemma, we represent the kernel $k_a^{\alpha,0}(|\xi|)$ as

$$k_a^{\alpha,0}(|\xi|) = \chi_A(|t|)\frac{a(|t|)(e^{i|t|}-1)}{|t|^{n-\alpha}} + \chi_A(|t|)\frac{a(|t|)-a(0)}{|t|^{n-\alpha}} + \frac{a(0)}{|t|^{n-\alpha}}$$
$$+ a(0)\frac{\chi_A(|t|)-1}{|t|^{n-\alpha}} \equiv k_{a,1}^{\alpha,0}(|t|) + k_{a,2}^{\alpha,0}(|t|) + k_{a,3}^{\alpha,0}(|t|) + k_{a,4}^{\alpha,0}(|t|).$$

Applying the Sobolev theorem, we obtain that the kernels $k_{a,1}^{\alpha,0}$ and $k_{a,2}^{\alpha,0}$ belong to L_p^q in some trapezium, containing the set $[O', O, A, A']$. Moreover, $k_{a,4}^{\alpha,0} \in L_p^q$ if $(1/p, 1/q) \in [A', A, E] \setminus (\{A'\} \cup \{A\})$. Since $k_{a,3}^{\alpha,0} \in L_p^q$ if and only if $(1/p, 1/q) \in (A', A)$, we obtain that the points, lying below the straight line $A'A$, do not belong to $\mathcal{L}(K_a^{\alpha,0})$.

To arrive at the relation $\{A\} \notin \mathcal{L}(K_a^{\alpha,0})$, we only have to apply Theorem 3.3 from [5] to the kernel $k_{a,3}^{\alpha,0}(|t|) + k_{a,4}^{\alpha,0}(|t|)$. Then $\{A'\} \notin \mathcal{L}(K_a^{\alpha,0})$ by duality. ∎

REMARK 5.1. We note that condition (1.2) is violated for the characteristic (1.3), if $0 < \beta < \alpha/n$. Indeed, $k_a^{\alpha,0} \in L_1^q$, $1 \le q < n/(n-\alpha)$ by Lemma 5.1. On the other hand, the application of Theorem 3.3 from [5] yields $k_a^{\alpha,0} \notin L_1^q$, if $1/(1-\beta) < q < n/(n-\alpha)$; we obtain a contradiction.

LEMMA 5.2. *The imbedding*

$$\mathcal{L}(K_a^{\alpha,\infty}) \supset \mathcal{L}_2(\alpha, n), \quad 0 < \alpha < n \tag{5.3}$$

is valid.

P r o o f. First, let $\alpha \neq 1, 2, \dots$ Applying the Taylor formula, we have

$$a(r) = \sum_{k=0}^{m-1} \frac{(a^*)^{(k)}(0)}{k! r^k} + R_m(1/r), \quad r > A, \quad m = [\alpha] + 1.$$

Correspondingly, $K_a^{\alpha,\infty}\varphi$ can be represented as follows

$$(K_a^{\alpha,\infty}\varphi)(x) = \sum_{k=0}^{m-1} \frac{(a^*)^{(k)}(0)}{k!} (S^{\alpha-k}\varphi)(x) + (T_m^\alpha \varphi)(x), \tag{5.4}$$

where $S^{\alpha-k}$ were defined in (4.1) and

$$(T_m^\alpha \varphi)(x) = \int\limits_{|t|>A} \frac{e^{i|t|} R_m(1/|t|)}{|t|^{n-\alpha}} \varphi(x-t)\, dt.$$

Since $\mathcal{L}(T_m^\alpha) = [O', O, E]$ (the kernel of T_m^α is bounded and belongs to L_1), the imbedding (5.3) follows from (4.2) for $0 < \alpha < 1$. In the case $\alpha > 1$ we also have (5.3) by virtue of (2.1) and (4.2).

Let $\alpha = \ell$, $\ell = 1, \dots, n-1$, then

$$(K_a^{\ell,\infty}\varphi)(x) = \sum_{k=0}^{\ell-1} \frac{(a^*)^{(k)}(0)}{k!} (S^{\ell-k}\varphi)(x) + \frac{(a^*)^{(\ell)}(0)}{\ell!} (S^0\varphi)(x) + (T_{\ell+1}\varphi)(x). \tag{5.5}$$

Imbedding (4.2) yields,

$$\mathcal{L}_2(\ell - k, n) \subset \mathcal{L}(S^{\ell-k}), \quad 0 \le k \le \ell - 1. \tag{5.6}$$

We also note that

$$\mathcal{L}(T^{\ell}_{\ell+1}) = [O', O, E]. \tag{5.7}$$

Besides this, the operator

$$(S^0\varphi)(x) = \int\limits_{|t|>A} \frac{e^{i|t|}\varphi(x-t)}{|t|^n}\,dt$$

is bounded from L_1 into L_q, $1 < q \le \infty$. Moreover, it is bounded in L_p, $1 < p < \infty$ (see [16, p. 394]). Interpolating, we obtain that

$$\mathcal{L}(S^0) \supset [O, O', E] \setminus (\{O'\} \cup \{O\}). \tag{5.8}$$

From (5.6)–(5.8) we obtain (5.3). ∎

Let us p r o v e T h e o r e m 2.1. In view of (5.1), the validity of (2.2) follows from Lemmas 5.1 and 5.2.

Passing to the proof of part II, we note that the statements 1) and 3) follow from the corresponding results of Theorem 4.1 in view of (5.4) and (5.5). Then 2) follows from 1) by duality.

To prove 4), we only have to note that the operator $S^{\frac{n-1}{2}}$ is not bounded in L_2, as is seen from (4.3), while all the other operators on the right–hand sides of (5.4) and (5.5) are bounded in this space.

Finally, 5) follows from the second statement of Lemma 5.1, since the points, lying below the straight line $A'A$, belong to $\mathcal{L}(K^{\alpha,\infty}_a)$. ∎

5.2. Proof of Theorem 5.2.

The proof is essentially based on the fact that we succeeded to construct the $\mathcal{L}$-characteristic of the operator M^{β}_u with the kernel

$$\chi(|t|)u(|t|)(1 - |t|^2 + i0)^{\beta-1}, \quad 0 < \beta < 1,$$

where $u(r) = d(r)e^{ir}/r^{n-\alpha}$, $d(r)$ and $\chi(r)$ were described above (after equality (1.3)).

Denote

$$\Lambda(\beta, n) = [O', O, T, M, T'] \setminus (\{T'\} \cup \{T\}).$$

The following lemma plays a crucial role in the proof of Theorem 2.2.

Lemma 5.3. *Let $0 < \beta < 1$. Then*

$$\mathcal{L}(M^{\beta}_u) = \Lambda(\beta, n).$$

P r o o f. We have

$$(M_u^\beta \varphi)(x) = \left(\int\limits_{1/2<|t|<1} + \int\limits_{1<|t|<3/2} \right) u(|t|)(1-|t|+i0)^{\beta-1}\varphi(x-t)\,dt$$
$$\equiv (M_u^{\beta,1}\varphi)(x) + (M_u^{\beta,2}\varphi)(x).$$

The symbol of the operator $M_u^{\beta,1}$ can be written in the form

$$m^1_{\beta,u}(\xi) = \int\limits_{1/2}^{1} \rho^{n-1}(1-\rho)^{\beta-1}u(\rho)\,d\rho \int\limits_{S^{n-1}} e^{i\rho(\xi\cdot\sigma)}\,d\sigma.$$

Consider the correspondent multiplier operator

$$(N_u^{\beta,1}\varphi)(x) = (2\pi)^{-n} \int\limits_{\mathrm{R}^n} m^1_{\beta,u}(\xi)\hat{\varphi}(\xi)e^{-ix\cdot\xi}\,d\xi, \quad \varphi \in S.$$

Evidently,

$$(M_u^{\beta,1}\varphi)(x) = (N_u^{\beta,1}\varphi)(x), \tag{5.9}$$

$\varphi \in S$. We prove that

$$\Lambda_1(\beta,n) \subset \mathcal{L}(N_u^{\beta,1}), \tag{5.10}$$

where

$$\Lambda_1(\beta,n) = \Lambda(\beta,n) \setminus (\{O'\} \cup \{O\})$$

(the imbedding (5.10) means, that the operator $M_u^{\beta,1}$ can be extended to a bounded operator from L_p into L_q, if $(1/p, 1/q) \in \Lambda_1(\beta,n)$).

Let $v(r) \in C^\infty(\mathrm{R}^1_+)$ be such that $v(r)=0$, if $r \le 1$, $v(r)=1$, if $r \ge 2$ and $0 \le v(r) \le 1$. Then

$$m^1_{\beta,u}(\xi) = (1-v(|\xi|^2))m^1_{\beta,u}(\xi) + v(|\xi|^2)m^1_{\beta,u}(\xi) \equiv m^{1,0}_{\beta,u}(\xi) + m^{1,\infty}_{\beta,u}(\xi).$$

It is evident that $m^{1,0}_{\beta,u}(\xi) \in M_p^q$, $1 \le p \le q \le \infty$; consider $m^{1,\infty}_{\beta,u}(\xi)$. The application of formula (25.13) from [12, p. 485] yields

$$m^{1,\infty}_{\beta,u}(\xi) = \frac{(2\pi)^{n/2}v(|\xi|^2)}{|\xi|^{\frac{n-2}{2}}} \int\limits_{1/2}^{1} \rho^{n/2}(1-\rho)^{\beta-1}u(\rho)J_{\frac{n-2}{2}}(\rho|\xi|)\,d\rho.$$

Making use of formula (3.1) with

$N = \left[\frac{n+1}{2}\right] + 2$, we obtain

$$m^{1,\infty}_{\beta,u}(\xi) = \sum_{k=0}^{N}(h_1^{\beta,k,-}(|\xi|) + h_1^{\beta,k,+}(|\xi|)) + R_1^{\beta,N,-}(|\xi|) + R_1^{\beta,N,+}(|\xi|), \quad (5.11)$$

where

$$h_1^{\beta,k,\pm}(|\xi|) = \frac{\gamma_{k,\pm} v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}+k}} \int_{1/2}^{1} (1-\rho)^{\beta-1} \rho^{\frac{n-1}{2}-k} u(\rho) e^{\pm i\rho|\xi|}\, d\rho,$$
$$0 \le k \le N, \quad \gamma_{0,\pm} = (2\pi)^{\frac{n-1}{2}} e^{\mp \frac{i\pi}{4}(n-1)},$$

$$R_1^{\beta,N,\pm}(|\xi|) = \frac{\gamma_{N+1,\pm} v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}}} \int_{1/2}^{1} (1-\rho)^{\beta-1} \rho^{\frac{n-1}{2}} u(\rho) e^{\pm i\rho|\xi|} R_{N,\pm}^{\left(\frac{n-2}{2}\right)}(\rho|\xi|)\, d\rho.$$

The application of Theorem 8.2 from [13] yields $R_1^{\beta,N,\pm}(|\xi|) \in \mathcal{R}_0$. Moreover, $R_1^{\beta,N,\pm}(|\xi|) \in L_1$, hence, $R_1^{\beta,N,\pm} \in M_p^q$, $1 \le p \le q \le \infty$.

Consider $h_1^{\beta,0,\pm}(|\xi|)$. After the change of variable $1-\rho = \tau$, we have

$$h_1^{\beta,0,\pm}(|\xi|) = \frac{\gamma_{0,\pm} e^{\pm i|\xi|} v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}}} \int_0^{1/2} \tau^{\beta-1}(1-\tau)^{\frac{n-1}{2}} u(1-\tau) e^{\mp i|\xi|\tau}\, d\tau.$$

Let $\omega(r) \in C^\infty(\mathrm{R}^1_+)$ be such that $0 \le \omega(r) \le 1$, $\omega(r) = 0$, if $r > 1/2$ and $\omega(r) = 1$, if $r < 1/4$. Then $h_1^{\beta,0,\pm}(|\xi|)$ can be represented as follows

$$h_1^{\beta,0,\pm}(|\xi|) = h_{1,1}^{\beta,0,\pm}(|\xi|) + h_{1,2}^{\beta,0}(|\xi|), \quad (5.12)$$

where

$$h_{1,1}^{\beta,0,\pm}(|\xi|) = \frac{\gamma_{0,\pm} e^{\pm i|\xi|} v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}}} \int_0^{1/2} \tau^{\beta-1}(1-\tau)^{\frac{n-1}{2}} u(1-\tau) e^{\mp i|\xi|\tau}(1-\omega(\tau))\, d\tau, \quad (5.13)$$

$$h_{1,2}^{\beta,0,\pm}(|\xi|) = \frac{\gamma_{0,\pm} e^{\pm i|\xi|} v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}}} \int_0^{1/2} \tau^{\beta-1}(1-\tau)^{\frac{n-1}{2}} u(1-\tau) e^{\mp i|\xi|\tau} \omega(\tau)\, d\tau. \quad (5.14)$$

We rewrite (5.14) in the form

$$h_{1,2}^{\beta,0,\pm}(|\xi|) = K^{\pm}(|\xi|)\frac{\gamma_{0,\pm}v(|\xi|^2)e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}+\beta}}, \tag{5.15}$$

where

$$K^{\pm}(|\xi|) = |\xi|^{\beta}\int\limits_0^{1/2}\tau^{\beta-1}(1-\tau)^{\frac{n-1}{2}}u(1-\tau)e^{\mp i|\xi|\tau}\chi(\tau)\,d\tau.$$

The second factor in the right-hand side of (5.15) belongs to M_p^q, if $(1/p,1/q) \in \Lambda(\beta,n)$, in accordance with Theorem 3.2. Moreover, $K^{\pm}(|\xi|) \in M_p^p$, $1 < p < \infty$ by the Mikhlin theorem. Then $h_{1,2}^{\beta,0,\pm}(|\xi|) \in M_p^q$, if $(1/p,1/q) \in \Lambda_1(\beta,n)$.

As regards multiplier (5.13), after integrating by parts, we represent it as a sum of the following terms:

$$\frac{\sigma_s^{\pm}v(|\xi|^2)e^{\pm\frac{i|\xi|}{2}}}{|\xi|^{\frac{n-1}{2}+s}}, \quad 1 \le s \le [n/2]+1, \tag{5.16}$$

$$\frac{\nu^{\pm}v(|\xi|^2)e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}+[\frac{n}{2}]+1}}\int\limits_{1/4}^{1/2}g(\tau)e^{\mp i|\xi|\tau}\,d\tau, \tag{5.17}$$

where $g(\tau) = (\tau^{\beta-1}(1-\tau)^{\frac{n-1}{2}}u(1-\tau))^{([n/2]+1)}$, $\sigma_s^{\pm}$ and $\nu^{\pm}$ being some constants. Applying Theorem 3.2 to multiplier (5.16), we obtain that it belongs to M_p^q, if $(1/p,1/q) \in [O',O,E]$. As is easily verified, the function (5.17) belongs to $\mathcal{R}_0 \cap L_p$, $1 < p < \infty$, hence, it is in M_p^q, $1 \le p \le q \le \infty$, except for the case when $p = 1$ and $q = \infty$. From here we derive that $h_{1,1}^{\beta,0,\pm} \in M_p^q$, if $1 \le p \le q \le \infty$, except for the mentioned case.

Then $h_1^{\beta,0,\pm}(|\xi|) \in M_p^q$, if $(1/p,1/q) \in \Lambda_1(\beta,n)$, by (5.12). Besides this, $h_1^{\beta,k,\pm}(|\xi|) \in M_p^q$, if $(1/p,1/q) \in [O',O,E] \setminus (\{O'\} \cup \{O\} \cup \{E\})$ for $1 \le k \le N$.

Gathering the results, obtained for the multipliers on the right–hand side of (5.11), we have

$$m_{\beta,u}^{1,\infty}(|\xi|) \in M_p^q, \quad \text{if} \quad (1/p,1/q) \in \Lambda_1(\beta,n);$$

the imbedding (5.10) has been proved.

The equality (5.9) is extended by boundedness to the whole space L_p, $1 < p < \infty$, since the operators in both sides of (5.9) are bounded in L_p for such p. Hence,

$$\Lambda_1(\beta,n)\subset\mathcal{L}(M_u^{\beta,1}). \tag{5.18}$$

We consider the operator $M_u^{\beta,2}$. Its symbol $m^2_{\beta,u}(|\xi|)$ can be represented in the form

$$\begin{aligned} m^2_{\beta,u}(|\xi|) = m^{2,0}_{\beta,u} + \sum_{k=1}^{N}(h_2^{\beta,k,+}(|\xi|) + h_2^{\beta,k,-}(|\xi|)) + h_{2,1}^{\beta,0,+}(|\xi|) + h_{2,1}^{\beta,0,-}(|\xi|) \\ + h_{2,2}^{\beta,0,+}(|\xi|) + h_{2,2}^{\beta,0,-}(|\xi|) + R_2^{\beta,N,+}(|\xi|) + R_2^{\beta,N,-}(|\xi|), \end{aligned}$$

where

$$m^{2,0}_{\beta,u}(|\xi|) = (v(|\xi|^2)-1)e^{i\pi\beta}\int_1^{3/2}\rho^{n-1}(\rho-1)^{\beta-1}u(\rho)\,d\rho\int_{S^{n-1}}e^{i\rho(\xi\cdot\sigma)}\,d\sigma, \tag{5.19}$$

$$\begin{aligned} h_2^{\beta,k,\pm}(|\xi|) &= -e^{i\pi\beta}\frac{\gamma_{k,\pm}v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}+k}} \\ &\quad\times\int_1^{3/2}(\rho-1)^{\beta-1}\rho^{\frac{n-1}{2}-k}u(\rho)e^{\pm i\rho|\xi|}\,d\rho, \quad 1\le k\le N, \end{aligned} \tag{5.20}$$

$$\begin{aligned} h_{2,1}^{\beta,0,\pm}(|\xi|) &= -e^{i\pi\beta}\frac{\gamma_{0,\pm}v(|\xi|^2)e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}}} \\ &\quad\times\int_0^{1/2}\tau^{\beta-1}(1+\tau)^{\frac{n-1}{2}}u(1+\tau)e^{\pm i|\xi|\tau}(1-\omega(\tau))\,d\tau, \end{aligned} \tag{5.21}$$

$$\begin{aligned} h_{2,2}^{\beta,0,\pm}(|\xi|) &= -e^{i\pi\beta}\frac{\gamma_{0,\pm}v(|\xi|^2)e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}}} \\ &\quad\times\int_0^{1/2}\tau^{\beta-1}(1+\tau)^{\frac{n-1}{2}}u(1+\tau)e^{\pm i|\xi|\tau}\omega(\tau)\,d\tau, \end{aligned} \tag{5.22}$$

$$\begin{aligned} R_2^{\beta,N,\pm}(|\xi|) &= -e^{i\pi\beta}\frac{\gamma_{N+1,\pm}v(|\xi|^2)}{|\xi|^{\frac{n-1}{2}}} \\ &\quad\times\int_1^{3/2}(\rho-1)^{\beta-1}\rho^{\frac{n-1}{2}}u(\rho)e^{\pm i\rho|\xi|}R_{N,\pm}^{\left(\frac{n-2}{2}\right)}(\rho|\xi|)\,d\rho. \end{aligned} \tag{5.23}$$

Analogous consideration of multipliers (5.19)–(5.23) leads to the imbedding

$$\Lambda_1(\beta, n) \subset \mathcal{L}(N_u^{\beta,2}),$$

where $N_u^{\beta,2}$ is the multiplier operator with the symbol $m^2_{\beta,u}(|\xi|)$. As above, we have

$$\Lambda_1(\beta, n) \subset \mathcal{L}(M_u^{\beta,2}). \tag{5.24}$$

From (5.18) and (5.24) and the evident relations $\{O'\} \in \mathcal{L}(M_u^\beta)$, $\{O\} \in \mathcal{L}(M_u^\beta)$ we derive that

$$\Lambda(\beta, n) \subset \mathcal{L}(M_u^\beta). \tag{5.25}$$

Let us prove sharpness of this result, that is, let us prove that

$$\mathcal{L}(M_u^\beta) = \Lambda(\beta, n). \tag{5.26}$$

To prove this, we have to verify that the points, lying below the straight lines $n/q = 1/p - \beta$ and $1/q = n/p - n - \beta + 1$, as well as the points $\{T'\}$ and $\{T\}$, do not belong to $\mathcal{L}(N_u^\beta)$ (hence, they do not belong to $\mathcal{L}(M_u^\beta)$). Direct analysis of the proof of the imbedding

$$\Lambda_1(\beta, n) \subset \mathcal{L}(N_u^\beta)$$

shows, that the picture of boundedness of the operator N_u^β is determined by the multiplier

$$\mu_\beta(|\xi|) = h_{1,2}^{\beta,0,+}(|\xi|) + h_{1,2}^{\beta,0,-}(|\xi|) + h_{2,2}^{\beta,0,+}(|\xi|) + h_{2,2}^{\beta,0,-}(|\xi|).$$

This means that $\mu_\beta(|\xi|) \in M_p^q$ if $(1/p, 1/q) \in \Lambda_1(\beta, n)$, while all other multipliers, considered above, belong to M_p^q, if $(1/p, 1/q) \in [O', O, E] \setminus (\{O'\} \cup \{O\} \cup \{E\})$.

Thus, it suffices to prove sharpness of the result for this multiplier. Keeping in mind (3.2), we represent $\mu_\beta^\pm(|\xi|) \equiv h_{1,2}^{\beta,0,\pm}(|\xi|) + h_{2,2}^{\beta,0,\pm}(|\xi|)$ as follows

$$\begin{aligned}\mu_\beta^\pm(|\xi|) = \frac{\gamma_{0,\pm} v(|\xi|^2) e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}+\beta+1}} \Bigg[&|\xi|^{\beta+1} \left(\int_0^{1/2} \tau^{\beta-1} f_1(\tau) e^{\mp i|\xi|\tau} \, d\tau - \frac{a_{0,\beta}^\mp}{|\xi|^\beta} \right) \\ &- e^{-i\pi\beta} |\xi|^{\beta+1} \left(\int_0^{1/2} \tau^{\beta+1} f_2(\tau) e^{\pm i|\xi|\tau} \, d\tau - \frac{a_{0,\beta}^\pm}{|\xi|^\beta} \right) \Bigg] \\ &+ \gamma_{0,\pm} (a_{0,\beta}^\mp - a_{0,\beta}^\pm e^{-i\pi\beta}) \frac{v(|\xi|^2) e^{\pm i|\xi|}}{|\xi|^{\frac{n-1}{2}+\beta}},\end{aligned} \tag{5.27}$$

where $f_1(\tau) = (1-\tau)^{\frac{n-1}{2}} u(1-\tau)\omega(\tau)$, $f_2(\tau) = (1+\tau)^{\frac{n-1}{2}} u(1+\tau)\omega(\tau)$, $a^{\pm}_{0,\beta} = u(1)(\pm i)^{\beta}\Gamma(\beta) \neq 0$. Applying Corollary 3.1 to the expression in the brackets, we obtain that it belongs to M^p_p, $1 < p < \infty$. Besides this, $\gamma^{\pm}_0 v(|\xi|^2)e^{\pm i|\xi|}/|\xi|^{\frac{n-1}{2}+\beta+1} \in M^q_p$, if $(1/p, 1/q) \in [O', O, E]$ by Theorem 3.2. Hence, the first term on the right-hand side of (5.27) is in M^q_p, if $(1/p, 1/q) \in [O', O, E] \setminus (\{O'\} \cup \{O\} \cup \{E\})$.

As for the second term, we note first that $a^{-}_{0,\beta} - a^{+}_{0,\beta} e^{-i\pi\beta} = 0$. Applying Theorem 3.2 to the multiplier

$$\frac{a(\beta)v(|\xi|^2)e^{-i|\xi|}}{|\xi|^{\frac{n-1}{2}+\beta}},$$

where $a(\beta) = u(1)(2\pi)^{\frac{n-1}{2}}\Gamma(\beta)e^{-\frac{i\pi}{4}(n-1)}\left(e^{\frac{i\pi\beta}{2}} - e^{-\frac{3}{2}i\pi\beta}\right) \neq 0$, we come to the desired conclusion. $\blacksquare$

Returning to the proof of Theorem 2.2, we note that the statement I of this theorem follows from Theorem 2.1 and Lemma 2.3.

Let us prove II. To prove 1), we may use just the same counter–example as in the proof of corresponding statement of Theorem 4.1. Then 2) follows from 1) by duality. As for 3)–7), they are derived from the statement II of Theorem 2.1 (items 1)–3), 5)) and Lemma 5.3 by convexity and duality. Finally, 8) follows from the statement II (item 4)) of Theorem 2.1 and the fact, that the operator M^{β}_u is bounded in L_p, $1 \leq p \leq \infty$. $\blacksquare$

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